Pismeni ispit iz predmeta **Linearna algebra, 18.01.2013.**

1. Neka je $\mathcal{P}_2$ vektorski prostor svih realnih polinoma stepena ≤ 2,

$$\mathcal{P}_2 = \{ax^2 + bx + c \,|\, a, b, c \in \mathbb{R}\}.$$

a) Provjeriti da li je sa $\langle p, q\rangle = p(1)q(1) + 2p(0)q(0) + p(-1)q(-1)$ definiran unutrašnji (skalarni) proizvod na $\mathcal{P}_2$.
b) Za podprostor $\mathcal{L} \subseteq \mathcal{P}_2$ generisan polinomima $p_1(x) = 1$ i $p_2(x) = x$ odredite ortogonalni komplement.
c) Odredite ortogonalnu projekciju od $p(x) = -2x^2 + x + 2$ na $\mathcal{L}$.

2. Dat je vektorski podprostor $\mathcal{M}$ prostora $\mathbb{R}^4$ definisan sa

$$\mathcal{M} = \{(z_1, z_2, z_3, z_4)^\top \in \mathbb{R}^4 \,|\, z_1 + 2z_2 + z_3 = 0,\ 2z_1 + z_2 - z_3 = 0, z_1 + 5z_2 + 4z_3 = 0\}.$$

Odrediti mu jedan (direktni) komplement (koji nije ortogonalni komplement).

3. Dat je vektorski prostor $\mathcal{L}$ vektorskog prostora $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ definisan sa

$$\mathcal{L} = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \,|\, AX - XA = \mathbf{0}, X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\}.$$

Razmatrajući standardni unutrašnji proizvod za matrice $\langle A, B\rangle = trag(A^\top B)$ odrediti ortonormiranu bazu za $\mathcal{L}$.

4. Odrediti URV faktorizaciju matrice $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 5 \end{bmatrix}$.

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(a) Provjeriti da li je sa

$$\langle p, q \rangle = p(1)q(1) + 2p(0)q(0) + p(-1)q(-1)$$

definiran unutrašnji (skalarni) proizvod na $\mathcal{P}_2$.

b) Za podprostor $\mathcal{L} \subseteq \mathcal{P}_2$ generisan polinomima $p_1(x) = 1$ i $p_2(x) = x$ odredite ortogonalni komplement.

c) Odrediti ortogonalnu projekciju od $p(x) = -2x^2 + x + 2$ na $\mathcal{L}$.

R:

(a) Prisjetimo se

Unutrašnji proizvod na realnom (ili kompleksnom) vektorskom prostoru V je f-ja koja preslikava svaki uređen par vektora x, y u realni (ili kompleksan) skalar $\langle x, y \rangle$ tako da vrijede sljedeće četri osobine

- $\langle x, x \rangle$ je realan sa $\langle x, x \rangle \geq 0$, i $\langle x, x \rangle = 0$ akko $x = 0$,
- $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$ za svaki skalar α
- $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (za realni prostor, ovo postaje $\langle x, y \rangle = \langle y, x \rangle$)

Realan ili kompleksan vektorski prostor koji je opremljen sa unutrašnjim proizvodom zovemo unitarni prostor

(i) pokažimo da je $\langle p, p \rangle \in \mathbb{R}$, $\langle p, p \rangle \geq 0$ i $\langle p, p \rangle = 0$ akko $p = 0$

$$\langle p, p \rangle = p(1)p(1) + 2p(0)p(0) + p(-1)p(-1) =$$

$$= \underbrace{(a+b+c)^2}_{\in \mathbb{R}} + 2\underbrace{c^2}_{\in \mathbb{R}} + \underbrace{(a-b+c)^2}_{\in \mathbb{R}} \geq 0$$

$\langle p,p\rangle = 0$ akko $(a+b+c)^2 + 2c^2 + (a-b+c)^2 = 0$ akko

$$\begin{matrix}(a+b+c)^2=0\\ 2c^2=0\\ (a-b+c)^2=0\end{matrix} \iff \begin{matrix}a+b+c=0\\ c=0\\ a-b+c=0\end{matrix} \iff \begin{matrix}a+b=0\\ a-b=0\end{matrix}$$

$\iff a=0,\ b=0,\ c=0 \iff p=0$

vrijedi prva osobina

(ii) $\langle p, \alpha q\rangle = p(1)\alpha q(1) + 2p(0)\alpha q(0) + p(-1)\alpha q(-1) =$

$= \alpha\left(p(1)q(1) + 2p(0)q(0) + p(-1)q(-1)\right) = \alpha\langle p,q\rangle \quad \forall\alpha$

vrijedi druga osobina

(iii)

$\langle p, q+r\rangle = \underbrace{p(1)[q(1)+r(1)]}_{\in\mathbb{R}} + \underbrace{2p(0)[q(0)+r(0)]}_{\in\mathbb{R}} + \underbrace{p(-1)[q(-1)+r(-1)]}_{\in\mathbb{R}}$

$= p(1)q(1) + 2p(0)q(0) + p(-1)q(-1) + p(1)r(1) + 2p(0)r(0) + p(-1)r(-1)$

$= \langle p,q\rangle + \langle p,r\rangle$ vrijedi treća osobina

(iv) $\langle p,q\rangle = \underbrace{p(1)q(1)}_{\in\mathbb{R}} + \underbrace{2p(0)q(0)}_{\in\mathbb{R}} + \underbrace{p(-1)q(-1)}_{\in\mathbb{R}} = q(1)p(1) + 2q(0)p(0) + q(-1)p(-1)$

$= \langle q,p\rangle$

vrijedi četvrta osobina

Dati proizvod jest unutrašnji proizvod na $\mathcal{P}_2$.

b) Prisjetimo se

Ortogonalni komplement

Za podskup M unitarnog prostora V, ortogonalni komplement $M^{\perp}$ od M je definisan kao skup svih vektora u V koji su ortogonalni na svaki vektor u M, Tj.

$$M^{\perp} = \{x \in V \mid \langle m, x \rangle = 0 \quad \forall m \in M\}$$

$$\mathcal{L} = \text{span}\{p_1(x), p_2(x)\} = \{\alpha_1 p_1(x) + \alpha_2 p_2(x) \mid \alpha_1, \alpha_2 \in R\}$$

$$= \{\alpha_1 + \alpha_2 x \mid \alpha_1, \alpha_2 \in R\}$$

$\mathcal{L}^{\perp} = \{ax^2 + bx + c \mid$ gdje su a, b i c realni brojevi koje trebamo odrediti$\}$

Pa izaberimo proizvoljne $p \in \mathcal{L}$ i $q \in \mathcal{L}^{\perp}$

$$\langle p, q \rangle = 0$$

$$(\alpha_1 + \alpha_2)(a + b + c) + 2\alpha_1 c + (\alpha_1 - \alpha_2)(a - b + c) = 0$$

$$2\alpha_1 a + 2\alpha_2 b + 4\alpha_1 c = 0 \quad /:2$$

$$\alpha_1 a + \alpha_2 b + 2\alpha_1 c = 0$$

Trebamo odrediti a, b i c tako da ova jednakost vrijedi za $\forall \alpha_1, \alpha_2 \in R$

Za $a = 2$, $b = 0$, $c = -1$ $\quad \alpha_1 a + \alpha_2 b + 2\alpha_1 c = 0 \quad \forall \alpha_1, \alpha_2$

Kako je $\dim(\mathcal{P}_2) = 3$ i $\dim(\mathcal{L}) = 2$ to je $\dim(\mathcal{L}^{\perp}) = 1$.

Primjetimo da ako za izaberemo proizvoljan realan broj β moramo imati $a = -2\beta$, $b = 0$ da bi jednakost

$\alpha_1 a + \alpha_2 b + 2\alpha_1 c = 0$ vrijedila za sve realne $\alpha_1, \alpha_2 \in \mathbb{R}$.

Prema tome

$$\mathcal{L}^\perp = \{-2\beta x^2 + \beta \mid \beta \in \mathbb{R}\} = \{(-2x^2+1)\beta \mid \beta \in \mathbb{R}\}$$
$$= \text{span}\{-2x^2+1\}.$$

Ako sa $p_3(x)$ označimo polinom $p_3(x) = -2x^2+1$, primijetimo da

$$\langle p_1(x), p_3(x)\rangle = \langle 1, -2x^2+1\rangle = 1\cdot(-1) + 2\cdot 1\cdot 1 + 1\cdot(-1) = 0$$

$$\langle p_2(x), p_3(x)\rangle = \langle x, -2x^2+1\rangle = 1\cdot(-1) + 2\cdot 0\cdot 1 + (-1)\cdot(-1) = 0$$

Ortogonalni komplement prostora $\mathcal{L}$ je $\mathcal{L}^\perp = \text{span}\{-2x^2+1\}$.

c) Prisjetimo se

Ortogonalna projekcija

Za $v \in V$ neka je $v = m + n$, gdje je $m \in M$ i $n \in M^\perp$.
Vektor m se zove ortogonalna projekcija od v na M.

Primijetimo da je $p(x) = \underbrace{x+1}_{\in \mathcal{L}} + \underbrace{(-2)x^2+1}_{\in \mathcal{L}^\perp}$ pa je

$x+1$ ortogonalna projekcija od $p(x) = -2x^2+x+1$ na $\mathcal{L}$.

(#) Dat je vektorski podprostor $\mathcal{M}$ prostora $\mathbb{R}^4$ definisan sa

$$\mathcal{M} = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; z_1 + 2z_2 + z_3 = 0,\; 2z_1 + z_2 - z_3 = 0,\; z_1 + 5z_2 + 4z_3 = 0 \right\}.$$

Odrediti mu jedan (direktni) komplement (koji nije ortogonalni komplement).

R.: Primjetimo da

$$\mathcal{M} = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} z_1 + \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} z_2 + \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} z_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \ker\Bigg(\underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 5 & 4 & 0 \end{bmatrix}}_{=A} \Bigg)$$

Prema tome $\mathcal{M} = \ker(A)$ gdje je $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 5 & 4 & 0 \end{bmatrix}$.

Prisjetimo se

Komplementarni podprostori

Podprostore $\mathcal{X}$, $\mathcal{Y}$ prostora $\mathcal{V}$ kažemo da su komplementarni kadgod je $\mathcal{V} = \mathcal{X} + \mathcal{Y}$ i $\mathcal{X} \cap \mathcal{Y} = 0$ i u tom slučaju kažemo da je $\mathcal{V}$ direktna suma od $\mathcal{X}$ i $\mathcal{Y}$, i ovo označavamo sa $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$.

$$\mathcal{X} + \mathcal{Y} := \{ x + y \mid x \in \mathcal{X} \text{ i } y \in \mathcal{Y} \}.$$

Ako $\mathcal{X}$ i $\mathcal{Y}$ imaju redom baze $\mathcal{B}_{\mathcal{X}}$ i $\mathcal{B}_{\mathcal{Y}}$ vrijedi sljedeće

$$\mathcal{V} = \mathcal{X} \oplus \mathcal{Y} \iff \forall v \in \mathcal{V} \ \exists!\, x \in \mathcal{X}, y \in \mathcal{Y} \ \ v = x + y \iff \mathcal{B}_{\mathcal{X}} \cap \mathcal{B}_{\mathcal{Y}} = \emptyset \text{ i } \mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}} \text{ je baza za } \mathcal{V}$$

Odredimo prvo bazu za $\mathcal{M}$ tj. bazu za $\ker(A)$.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 5 & 4 & 0 \end{bmatrix} \underset{III_v - I_v}{\overset{II_v + I_v(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \underset{III_v + II_v}{\overset{II_v : (-3)}{\sim}} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{I_v + II_v \cdot (-2)}{\sim}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E_A$$

$$Ax = 0 \iff E_A x = 0 \quad \text{gdje je } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= -x_3 \end{aligned}$$

$x_3 = s \quad x_4 = t$

$$x = \begin{pmatrix} s \\ -s \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} t$$

$$\mathcal{M} = \ker(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Baza za $\mathcal{M}$ je $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Da bi odredili komplement od $\mathcal{M}$ nadopunimo skup $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ do baze za $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{III_v - I_v}{\overset{II_v + I_v}{\sim}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{II_v \leftrightarrow IV_v}{\sim} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\underset{IV_v + III_v}{\overset{I_v + III_v}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \overset{III_v \cdot (-1)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & 0 & \textcircled{1} & 1 \end{bmatrix}$$

↑ ovo je matrica u reduciranom red ešelon obliku

(Direktni) komplement od M je

$$N = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

(#) Dat je vektorski podprostor $\mathcal{L}$ vektorskog prostora $Mat_{2\times 2}(\mathbb{R})$ definisan sa

$$\mathcal{L} = \left\{ A \in Mat_{2\times 2}(\mathbb{R}) \mid AX - XA = 0, \ X = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}.$$

Posmatrajući standardni unutrašnji proizvod za matrice

$$\langle A, B \rangle = trag(A^T B)$$

odrediti ortonormiranu bazu za $\mathcal{L}$.

Rj:

Da bi odredili ortonormiranu bazu za $\mathcal{L}$ prvo je potrebno pronaći bilo kakvu bazu za $\mathcal{L}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left.\begin{aligned} AX &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2b & a \\ 2d & c \end{bmatrix} \\ XA &= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 2a & 2b \end{bmatrix} \end{aligned}\right\} \Rightarrow AX - XA = \begin{bmatrix} 2b-c & a-d \\ 2d-2a & c-2b \end{bmatrix}$$

$$AX - XA = 0 \iff \begin{aligned} 2b - c &= 0 \\ a - d &= 0 \\ 2d - 2a &= 0 \\ c - 2b &= 0 \end{aligned} \iff \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}}_{=B} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix} \underset{III_v + II_v}{\overset{IV_v + I_v \cdot 2}{\sim}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} a - d &= 0 \\ 2b - c &= 0 \end{aligned}$$

$$a = d$$
$$b = \frac{c}{2} \qquad c = 2b$$

Odavdje sad nije teško vidjeti da se prostor $\mathcal{L}$ može napisati u obliku:

$$\mathcal{L} = \left\{ \begin{bmatrix} d & b \\ 2b & d \end{bmatrix} \middle| \, b, d \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \alpha + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \beta \, \middle| \, \alpha, \beta \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$$

Baza za $\mathcal{L}$ je $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\}$.

Sad iskoristimo Gram-Schmidtovu proceduru pa odredimo ortonormiranu bazu za $\mathcal{L}$.

<u>Klasični Gram-Schmidtov algoritam</u>

Za $k=1$: $u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i$

$u_k \leftarrow \dfrac{u_k}{\|u_k\|}$

$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\|X_1\| = \sqrt{\operatorname{trag}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)} = \sqrt{2}$

$U_1 \quad \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$U_2 \leftarrow X_2 - \langle U_1, X_2 \rangle U_1$

$U_2 \leftarrow \dfrac{U_2}{\|U_2\|}$

$\langle U_1, X_2 \rangle = \operatorname{trag}(U_1^T X) = \operatorname{trag}([$

$= \operatorname{trag}\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right) = 0$

$X_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

$$U_2 \leftarrow X_2 - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\| U_2 \| = \sqrt{\operatorname{trag}\left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right)} = \sqrt{4+1} = \sqrt{5}$$

Ortonormirana baza za $\mathcal{L}$ je

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}.$$

Odrediti URV faktorizaciju matrice A

$$A=\begin{pmatrix}1 & 2 & 3\\ -1 & -2 & -3\\ 2 & 4 & 5\end{pmatrix}$$

Rj.

Prisjetimo se

URV faktorizacija

Za svaku matricu $A \in Mat_{m\times n}(\mathbb{R})$ ranga r, postoje ortogonalne matrice $U_{m\times m}$ i $V_{n\times n}$ i nesingularna matrica $C_{r\times r}$ takve da

$$A = URV^T = U\begin{pmatrix}C_{r\times r} & 0\\ 0 & 0\end{pmatrix}_{m\times n} V^T$$

- Prvih r kolona u U je ortonormirana baza za $im(A)$.
- Zadnjih $m-r$ kolona od U je ortonormirana baza za $ker(A^T)$.
- Prvih r kolona od V je ortonorm. baza za $im(A^T)$
- Zadnjih $n-r$ kolona od V je orton. baz za $ker(A)$

Pa prvo odredimo baze za četri fundamentalna podprostora.

$$A=\begin{pmatrix}1 & 2 & 3\\ -1 & -2 & -3\\ 2 & 4 & 5\end{pmatrix} \underset{III_v+I_v\cdot(-2)}{\overset{II_v+I_v}{\sim}} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \underset{I_v+III_v\cdot 3}{\overset{II_v\leftrightarrow III_v}{\sim}} \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix} \overset{II_v\cdot(-1)}{\sim} \overbrace{\begin{pmatrix}\textcircled{1} & 2 & 0\\ 0 & 0 & \textcircled{1}\\ 0 & 0 & 0\end{pmatrix}}^{E_A}$$

$$im(A)=span\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix},\begin{pmatrix}3\\-3\\5\end{pmatrix}\right\}, \quad im(A^T)=span\left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$Ax=0 \iff E_A x = 0$

$$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2 \qquad x = \begin{pmatrix} -2s \\ s \\ 0 \end{pmatrix},\ s \in \mathbb{R}$$
$$x_3 = 0$$

$$\ker(A) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -2 & -3 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right] \underset{III_v + I_v \cdot (-2)}{\overset{II_v + I_v}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \underset{III_v \cdot (-1)}{\overset{I_v + III_v \cdot (3)}{\sim}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \overset{II_v \leftrightarrow III_v}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underbrace{\quad}_{E_A} \qquad \underbrace{\quad}_{P}$$

$$\ker(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Sad uz pomoć Gram-Schmidtovog procesa ortonormirajmo date baze. Prisjetimo se

Klasični Gram-Schmidtov algoritam

Za $k=1$: $u_1 \leftarrow \frac{x_1}{\|x_1\|}$

Za $k>1$: $u_k \leftarrow x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

Posmatrajmo bazu za $\text{im}(A)$: $\left\{ \overset{x_1}{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}, \overset{x_2}{\begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix}} \right\}$

$$\|x_1\| = \sqrt{x_1^T x_1} = \sqrt{1+1+4} = \sqrt{6}$$

$$u_1 \leftarrow \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad u_2 \leftarrow x_2 - \langle u_1, x_2 \rangle u_1$$

$$\langle u_1, x_2 \rangle = u_1^T x_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{6}} (3+3+10) = \frac{16}{\sqrt{6}}$$

$$\langle u_1 x_2 \rangle u_1 = \frac{16}{6} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{8}{3} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$u_2 \leftarrow \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix} - \frac{8}{3} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 9-8 \\ -9+8 \\ 15-16 \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\|u_2\| = \sqrt{\frac{1}{9}(1+1+1)} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

$$u_2 \leftarrow \frac{\sqrt{3}}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Ortonormirana baza za $im(A)$ je $\left\{ \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}, \begin{pmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ -\sqrt{3}/3 \end{pmatrix} \right\}$

Posmatrajmo sad bazu za $im(A^T)$: $\left\{ \overset{x_1}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}, \overset{x_2}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$.

$$\|x_1\| = \sqrt{1+4} = \sqrt{5}$$

$$u_1 \leftarrow \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\langle u_1, x_2 \rangle = \frac{1}{\sqrt{5}} (1\ 2\ 0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$u_2 \leftarrow x_2 - \langle u_1, x_2 \rangle u_1$$

$$u_2 \leftarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\|u_2\| = 1$$

$$u_2 \leftarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Ortonormirana baza za $im(A^T)$ je $\left\{ \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Ortonormirane baze za $ker(A)$ i $ker(A^T)$ su redom

$$\left\{ \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix} \right\} \quad i \quad \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\}.$$

Time smo odredili matrice U i V

$$U = \begin{pmatrix} 1/\sqrt{6} & \sqrt{3}/3 & 1/\sqrt{2} \\ -1/\sqrt{6} & -\sqrt{3}/3 & 1/\sqrt{2} \\ 2/\sqrt{6} & -\sqrt{3}/3 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$A = URV^T \quad / \cdot U^T \text{ sa lijeve strane}$$

$$U^T A = RV^T \quad / \cdot V \text{ sa desne strane}$$

$$R = U^T A V$$

$$R = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ \sqrt{3}/3 & -\sqrt{3}/3 & -\sqrt{3}/3 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ \sqrt{3}/3 & -\sqrt{3}/2 & -\sqrt{3}/3 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 5/\sqrt{5} & 3 & 0 \\ -5/\sqrt{5} & -3 & 0 \\ 10/\sqrt{5} & 5 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 30/\sqrt{30} & 16/\sqrt{6} & 0 \\ 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$